

Problem 2.28

[Difficulty: 4]

2.28 Consider the velocity field of Problem 2.20. Plot the streakline formed by particles that passed through the point (1, 1) during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0, 1$, and 2 s.

Given: Velocity field

Find: Plot of streakline for $t = 0$ to 3 s at point (1,1); compare to streamlines through same point at the instants $t = 0, 1$ and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t , where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = B \cdot x \cdot (1 + A \cdot t) \quad A = 0.5 \quad \frac{1}{s} \quad B = 1 \quad \frac{1}{s} \quad v_p = \frac{dy}{dt} = C \cdot y \quad C = 1 \quad \frac{1}{s}$

So, separating variables $\frac{dx}{x} = B \cdot (1 + A \cdot t) \cdot dt \quad \frac{dy}{y} = C \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right) \quad \ln\left(\frac{y}{y_0}\right) = C \cdot (t - t_0)$

$$x = x_0 \cdot e^{B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right)} \quad y = y_0 \cdot e^{C \cdot (t - t_0)}$$

The pathlines are $x_p(t) = x_0 \cdot e^{B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right)} \quad y_p(t) = y_0 \cdot e^{C \cdot (t - t_0)}$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then $x_{st}(t_0) = x_0 \cdot e^{B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right)} \quad y_{st}(t_0) = y_0 \cdot e^{C \cdot (t - t_0)}$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{C \cdot y}{B \cdot x \cdot (1 + A \cdot t)}$$

So, separating variables
$$(1 + A \cdot t) \cdot \frac{dy}{y} = \frac{C}{B} \cdot \frac{dx}{x} \quad \text{which we can integrate for any given } t \text{ (} t \text{ is treated as a constant)}$$

Integrating
$$(1 + A \cdot t) \cdot \ln(y) = \frac{C}{B} \cdot \ln(x) + \text{const}$$

The solution is
$$y^{1+A \cdot t} = \text{const} \cdot x^{\frac{C}{B}}$$

For particles at (1,1) at $t = 0, 1$, and 2s $y = x$ $y = x^{\frac{2}{3}}$ $y = x^{\frac{1}{2}}$

